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Stealth Scalar Field Overflying a $2+1$ Black Hole

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A nontrivial scalar field configuration of vanishing energy–momentum is reported. These matter configurations have no influence on the metric and therefore they are not be "detected" gravitationally. This phenomenon occurs for a time–dependent nonminimally coupled and self–interacting scalar field on the $2+1$ (BTZ) black hole geometry. We conclude that such stealth configurations exist for the static $2+1$ black hole for any value of the nonminimal coupling parameter $\zeta \neq 0$ with a fixed self–interaction potential $U_\zeta(\Phi)$. For the range $0 < \zeta \leq 1/2$ potentials are bounded from below and for the range $0 < \zeta < 1/4$ the stealth field falls into the black hole and is swallowed by it at an exponential rate, without any consequence for the black hole.

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One of the hallmarks of General Relativity is the fact that the presence of matter can be detected by the metric. For instance, the total mass–energy contained in any form of matter that falls into a black hole can be revealed by the metric at infinity through the famous ADM formula. More generally, the role of matter in gravity is to provide a source for the curvature of spacetime through the energy–momentum tensor. This general feature can have exceptions, as we report here, if the nontrivial field content and the background geometry are so special that the energy–momentum tensor vanishes identically and therefore, the spacetime geometry can be exactly the same as the one that solves the matter–free Einstein equations.

Alberto García, for one, has made it a profession of dressing spacetime geometries with all sorts of exotic and nonstandard drapery. It is only fitting therefore, to contribute to a volume in his honor this stealth form of matter, which would certainly not escape his detection.

Consider a self–interacting scalar field nonminimally coupled to $2+1$ gravity described by the action [1]

$$S = \int d^3x\sqrt{-g}\left(\frac{1}{2\kappa}\left(R+2l^{-2}\right) - \frac{1}{2}\nabla_\mu\Psi\nabla^\mu\Psi - \frac{1}{2}\zeta\, R\,\Psi^2 - U(\Psi)\right), \tag{1}$$

where $\Lambda = -l^{-2}$ is the cosmological constant, ζ is the nonminimal coupling parameter and $U(\Psi)$ is the self–interaction potential. The corresponding field equations are

$$G_\mu^{\ \nu} - l^{-2}\delta_\mu^{\ \nu} = \kappa T_\mu^{\ \nu}, \tag{2}$$

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and

$$\Box\Psi = \zeta\, R\,\Psi + \frac{\mathrm{d}U(\Psi)}{\mathrm{d}\Psi}, \tag{3}$$

where the energy–momentum tensor is given by

$$T_\mu{}^\nu = \nabla_\mu\Psi\nabla^\nu\Psi - \delta_\mu{}^\nu\left(\frac{1}{2}\nabla_\alpha\Psi\nabla^\alpha\Psi + U(\Psi)\right) + \zeta\left(\delta_\mu{}^\nu\Box - \nabla_\mu\nabla^\nu + G_\mu{}^\nu\right)\Psi^2. \tag{4}$$

We are interested in gravitationally undetectable configurations in the sense defined in the first paragraph, i.e., nontrivial solutions to the field equations (2) and (3) such that $T_\mu{}^\nu = 0$. For these configurations both sides of Einstein equations vanish independently. The only symmetry we shall impose is that the solutions possess cyclic symmetry, that is, they must be invariant under the action of the 1–parameter group $SO(2)$. Under these conditions the metric must be a cyclic solution of the $2+1$ vacuum Einstein equations with a negative cosmological constant,

$$G_\mu{}^\nu - l^{-2}\delta_\mu{}^\nu = 0.$$

By Birkhoff's theorem in $2+1$ dimensions [2] the geometry must be given by the so–called BTZ black hole solution [3, 4][1]

$$\boldsymbol{g}_{\mathrm{BTZ}} = -F(r)\boldsymbol{dt}^2 + \frac{\boldsymbol{dr}^2}{F(r)} + r^2\left(\boldsymbol{d\phi} - \frac{J}{2r^2}\boldsymbol{dt}\right)^2, \tag{5}$$

with

$$F(r) \equiv \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \tag{6}$$

where M and J are the mass and angular momentum of the black hole, respectively, with $|J| \leq Ml$. In these coordinates the cyclic symmetry is generated by the Killing field $\boldsymbol{m} = \boldsymbol{\partial_\phi}$ and the cyclic invariance of the scalar field is expressed in the dependence $\Psi = \Psi(t, r)$.

In the present context a stealth configuration is a nontrivial solution Ψ such that $T_\mu{}^\nu(\boldsymbol{g}_{\mathrm{BTZ}}, \Psi) = 0$. Using definition (4), these equations can be explicitly written for the metric (5) as

$$T_t{}^t - T_\phi{}^\phi = \frac{\Psi\partial_t\Psi}{F}\left(2\zeta\frac{\partial^2_{tt}\Psi}{\partial_t\Psi} - (1-2\zeta)\frac{\partial_t\Psi}{\Psi} - 2\zeta\frac{MF}{r}\frac{\partial_r\Psi}{\partial_t\Psi}\right) = 0, \tag{7a}$$

$$T_t{}^r = -F\Psi\partial_t\Psi\left(2\zeta\frac{\partial^2_{rt}\Psi}{\partial_t\Psi} - (1-2\zeta)\frac{\partial_r\Psi}{\Psi} - 2\zeta\frac{r}{l^2F}\right) = 0, \tag{7b}$$

$$T_\phi{}^t = J\zeta\frac{\Psi\partial_r\Psi}{r} = 0, \tag{7c}$$

$$T_\phi{}^r = -J\zeta\frac{\Psi\partial_t\Psi}{r} = 0, \tag{7d}$$

$$T_\phi{}^\phi - T_r{}^r = F\Psi\partial_r\Psi\left(2\zeta\frac{\partial^2_{rr}\Psi}{\partial_r\Psi} - (1-2\zeta)\frac{\partial_r\Psi}{\Psi} + \frac{\zeta(4Mr^2 - J^2)}{2r^3F}\right) = 0, \tag{7e}$$

[1] We shall not consider here the self–dual Coussaert–Henneaux spacetimes [5], which are also allowed by Birkhoff's theorem (see [2]), since it can be shown that in this case there are no stealth configurations for which $T_\mu{}^\nu = 0$.

$$T_\phi{}^\phi - T_r{}^r - T_t{}^t = U(\Psi) - \frac{\Psi^2}{2}\left(F\frac{(\partial_r\Psi)^2}{\Psi^2} + \frac{8\zeta(r^2-Ml^2)}{l^2 r}\frac{\partial_r\Psi}{\Psi} - \frac{1}{F}\frac{(\partial_t\Psi)^2}{\Psi^2} + \frac{2\zeta}{l^2}\right) = 0. \quad \text{(7f)}$$

There is no need to solve the scalar equation (3) as it is automatically satisfied as a consequence of the conservation of the energy–momentum tensor (4). From Eqs. (7c) and (7d), one concludes that the existence of nontrivial configurations requires that the background geometry be a non–rotating black hole, $J = 0$. (Another possibility following from Eqs. (7c) and (7d) is that the scalar field be minimally coupled, $\zeta = 0$, but this condition would imply the trivial solution $\Psi(t, r) = \text{const.}$)

For $J = 0$ and $\zeta \neq 1/4$, Eq. (7e) can be straightforwardly integrated, giving

$$\Psi(t,r) = \left(f(t)\sqrt{r^2 - Ml^2} + h(t)\right)^{-2\zeta/(1-4\zeta)}, \tag{8}$$

where $f(t)$ and $h(t)$ are integration functions. This result is further restricted by Eqs. (7b) and (7a) giving $h(t) = h = \text{const.}$ and

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - \frac{M}{l^2} f = 0, \tag{9}$$

respectively. Hence, the final expression for the scalar field reads

$$\Psi(t,r) = \left[K\cosh\left(\frac{\sqrt{M}}{l}(t-t_0)\right)\sqrt{r^2 - Ml^2} + h\right]^{-2\zeta/(1-4\zeta)}, \tag{10}$$

for $M \neq 0$, and

$$\Psi(t,r) = \left[K(t-t_0)r + h\right]^{-2\zeta/(1-4\zeta)}, \tag{11}$$

for $M = 0$, where K, h, and t_0 are now integration constants. Inserting these expressions in Eq. (7f) and using Eqs. (10) and (11) once again, the generic self–interaction potential allowing the existence of a stealth configuration must be of the form

$$U_\zeta(\Psi) = \frac{\zeta\Psi^2}{l^2(1-4\zeta)^2}\left(2\zeta\lambda\,|\Psi|^{(1-4\zeta)/\zeta} + 4\zeta(1-8\zeta)h\,|\Psi|^{(1-4\zeta)/(2\zeta)} + (1-8\zeta)(1-6\zeta)\right), \tag{12}$$

where $\lambda = h^2 + K^2Ml^2$ for $M \neq 0$ and $\lambda = h^2 - K^2l^4$ for $M = 0$.

The above potential has three parameters: ζ, λ, and h. The solutions are characterized by two integration constants the mass M and t_0, which can be eliminated by a time translation. The constant K is clearly not an independent integration constant, but it is a function of the coupling constants appearing in the action and the black hole mass.

The case $\zeta = 1/8$ is exceptional because not only the nonminimal coupling is conformally invariant but the self–interaction potential reduces to $8l^2U_{1/8}(\Psi) = \lambda\Psi^6$ which is also conformally invariant in $2+1$ dimensions. In this case the solutions have an additional integration constant, h, which cannot be related to the coupling constants appearing in the action. The occurrence of this new integration constant could be related to the conformal invariance of the matter sector.

Examples of the above potentials for different nonminimal coupling are shown in FIG. 1.

We have shown that for any nonvanishing value of the nonminimal coupling parameter $\zeta \neq 1/4$, Eqs. (2) and (3) with self–interaction potential (12) have a nontrivial solution

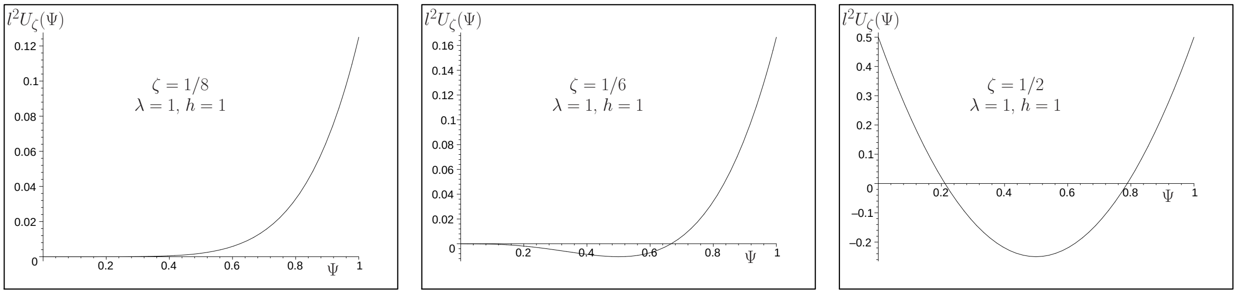

FIG. 1: The self–interaction potential (12) for conformal coupling ($\zeta = 1/8$), and for the nonminimal couplings $\zeta = 1/6$ and $\zeta = 1/2$ respectively. The coupling constants in the potential are fixed as $\lambda = 1$ and $h = 1$.

given by the static black hole [(5) with $J = 0$] for the geometry and by the time–dependent expression (10) for the scalar field.

For $\zeta = 1/4$ the expression (8) is ill defined, this is due to the fact that in this case Eq. (7e) is on a logarithmic branch, as can be seen from its first integral

$$\frac{\partial_r \Psi}{\Psi^{(1-2\zeta)/(2\zeta)}} = \frac{r\tilde{f}(t)}{\sqrt{r^2 - Ml^2}}. \tag{13}$$

Clearly, for $\zeta = 1/4$ the left hand side integrates as a logarithm, giving for the scalar field

$$\Psi(t,r) = \tilde{h}(t)\exp\left(\tilde{f}(t)\sqrt{r^2 - Ml^2}\right). \tag{14}$$

Using now Eqs. (7b) and (7a) evaluated for $\zeta = 1/4$ we conclude that $\tilde{h}(t)$ is constant and $\tilde{f}$ satisfies Eq. (9), as in the generic case. Hence, for $\zeta = 1/4$ the scalar field is

$$\Psi(t,r) = \Psi_0 \exp\left[K\cosh\left(\frac{\sqrt{M}}{l}(t - t_0)\right)\sqrt{r^2 - Ml^2}\right], \tag{15}$$

when $M \neq 0$, and

$$\Psi(t,r) = \Psi_0 \exp\left[K(t - t_0)r\right], \tag{16}$$

for $M = 0$. These expressions, together with Eq. (7f), imply a non–polynomial form for the self–interaction potential

$$U_{1/4}(\Psi) = \frac{\Psi^2}{2l^2}\left\{\left[\ln\left(\frac{\Psi}{\Psi_0}\right) + 1\right]^2 + \lambda_1 - \frac{1}{2}\right\}, \tag{17}$$

where $\lambda_1 = K^2Ml^2$ for $M \neq 0$ and $\lambda_1 = -K^2l^4$ for $M = 0$. This potential is shown in FIG. 2 for different values of the constant λ_1. It should be noticed that although this potential is bounded from below, the scalar field behaves explosively (as an exponential for $M = 0$ and as the exponential of an exponential for $M \neq 0$) for large times.

We are only interested in values of ζ which give rise to physically reasonable configurations. The values $\zeta < 0$ and $\zeta > 1/2$ should be discarded as they produce self–interaction potentials which are unbounded from below. On the other hand, for $1/4 \leq \zeta \leq 1/2$ the

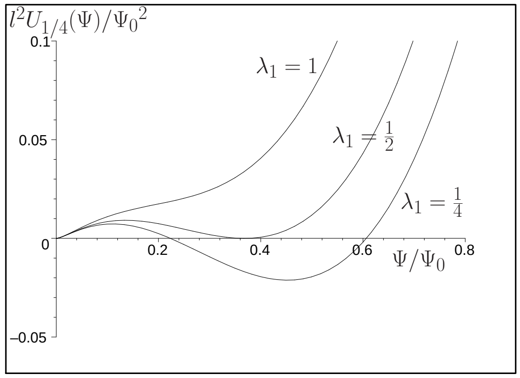

FIG. 2: Logarithmic self–interaction potential (17) for the nonminimal coupling $\zeta = 1/4$. The form of the potential is exhibited for different values of the constant λ_1.

scalar field presents an explosive growth in time. Hence, we restrict our analysis to the nonminimal coupling parameters lying in the range $0 < \zeta < 1/4$.

The physical interpretation of this solution for the black hole case $M \neq 0$ (in the range $0 < \zeta < 1/4$) can be better understood from its behavior in ingoing Eddington–Finkelstein coordinates $(v = t - t_0 + r^*, r, \phi)$, which for the static black hole are given by

$$v = t - t_0 + r^* = t - t_0 + \frac{l}{2\sqrt{M}} \ln\left(\frac{r - \sqrt{M}l}{r + \sqrt{M}l}\right). \tag{18}$$

In these coordinates, the scalar field is expressed as

$$\Psi(v, r) = \left\{K\left[r\cosh\left(\frac{\sqrt{M}}{l}v\right) + \sqrt{M}l\sinh\left(\frac{\sqrt{M}}{l}v\right)\right] + h\right\}^{-2\zeta/(1-4\zeta)}, \tag{19}$$

and in contrast with expression (10), it is evidently smooth at the horizon $r_+ = \sqrt{M}l$ for all times.[2] A graphic description of this expression, in the range $0 < \zeta < 1/4$, for different times starting from $v = 0$ is shown below in FIG. 3. From this it can be seen that a smooth initial scalar field configuration starts to fall into the black hole and is eventually swallowed by it.

We would like to stress the novel features of the stealth configurations in comparison with similar solutions presented previously (see [1, 6, 7, 8]). First, here a self–interaction potential is included. This circumvents the problem that for $U(\Psi) = 0$, Eq. (7f) would become a constraint severely restricting the parameters of the problem. In particular, equating to zero the right hand side of (12) implies that a nontrivial stealth solution would only exist if $\zeta = 1/8$ or $\zeta = 1/6$, and furthermore, only for $h = 0 = M$ [1]. Hence, the inclusion of the self–interaction allows to have stealth solutions for any value of the nonminimal coupling parameter, and for any black hole mass.

Another important feature of the stealth solutions is the time dependence since the solutions discussed in Refs. [1, 6, 7, 8] were stationary and they only exist for the zero mass BTZ geometry. This stationary solutions belong to a special class, which is obtained from Eq. (9)

[2] We thank Viqar Husain for helping us to elucidate this point.

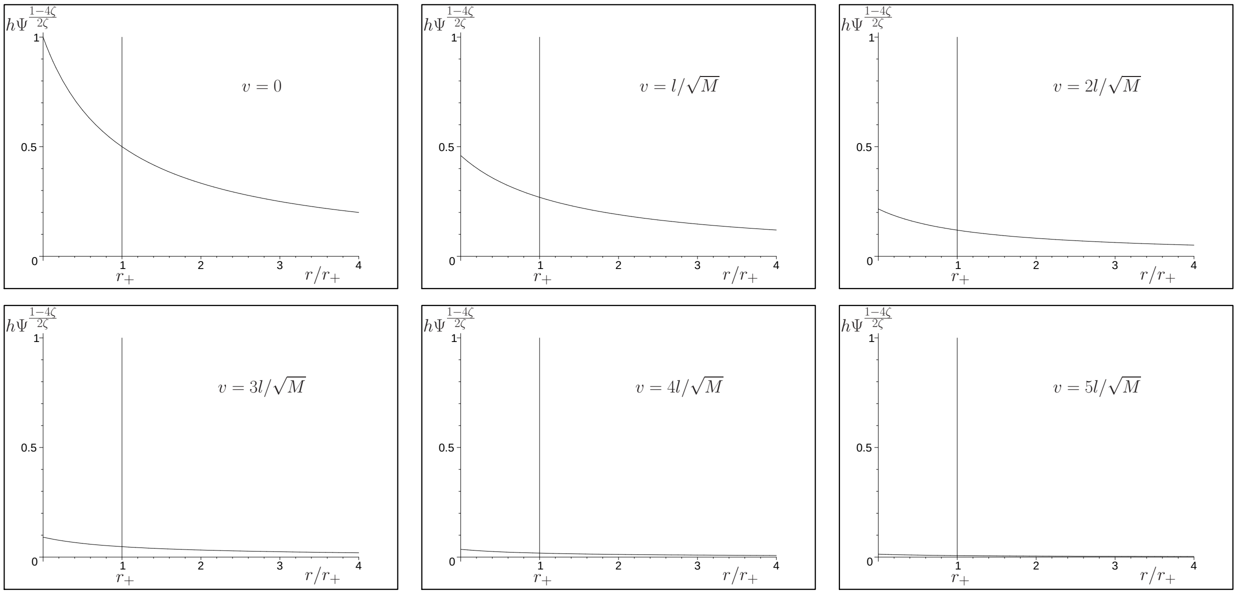

FIG. 3: Sequence exhibiting the evolution of the stealth field as time, v, increases for nonminimal couplings in the range $0 < \zeta < 1/4$. All the graphs were made for $K = h/\sqrt{M}l$.

for $f(t) = K =$ const. and $M = 0$. This is equivalent to taking the limit $M \to 0$ in Eqs. (10) and (15). The time dependence is strictly required in order to have stealth solutions when $M \neq 0$, as can be concluded from Eq. (7a). In relation with this, one can ask if allowing a nontrivial angular dependence we can also allow for nonzero angular momentum. The answer to this question, however, is negative. Including an angular dependence in Ψ makes the system $T_\mu{}^\nu(\boldsymbol{g}_{\mathrm{BTZ}}, \Psi) = 0$ although obviously quite involved, it can be integrated again. The condition that these solutions respect the identification $\phi = \phi + 2\pi$ globally implies that $\partial_\phi \Psi = 0$. Hence, expression (10) is the most general solution on the $2+1$ black hole.

The stealth solutions presented here have no influence on the gravitational field, but an important issue is whether their quantum fluctuations would produce back reaction on the geometry or not. The question is whether quantum corrections to the black hole would produce a nonzero expectation value of the quantum energy–momentum operator.

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